On Log Canonical Models of the Moduli Space of Stable Pointed Curves.

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We study the log canonical models of the moduli space $\overline{\mathbf{M}}_{0,n}$ of pointed stable genus zero curves associated to the divisors of the form $K_{\overline{\mathbf{M}}_{0,n}} + \alpha \mathrm{D}$ where D denotes the boundary of $\overline{\mathbf{M}}_{0,n}$. In particular we will show that, as a formal consequence of a conjecture by Fulton regarding the ample cone of $\overline{\mathbf{M}}_{0,n}$, the log canonical model associated to $K_{\overline{\mathbf{M}}_{0,n}} + \alpha \mathrm{D}$ is equal to the moduli space of weighted stable curves with symmetric weights dependant on α.

1 Introduction

Historically, an important problem related to the minimal model program is to find the canonical model for $\overline{\mathbf{M}}_g$, the moduli space of stable genus g curves. Here, the canonical model (if it exists) is the projective scheme associated to the graded ring (called the canonical ring) of sections $\bigoplus_{n\geq 0} \Gamma(nK_{\overline{\mathbf{M}}_g})$. $\overline{M}_g$ has been proven to be of general type for $g \geq 22$ [EH,Fa,HaMu]. In particular in this case the finite generation of the canonical ring implies that the canonical model is birational to $\overline{M}_g$. Recent progress has been made in this area. For example [BCHM] have proved the existence of canonical models in the case of smooth projective varieties of general type.

A somewhat easier problem is to study the *log* canonical model associated to the graded ring $\bigoplus_{n\geq 0} \Gamma(n(K_{\overline{\mathbf{M}}_g} + \alpha D))$ where D is the boundary component parameterizing singular curves and α is a rational number. This ring is known to be finitely generated for $7/10 \leq \alpha \leq 1$ and the models have been explicitly constructed using geometric invariant theory for certain values of α [CoHa,HH].

We play a similar game with the related but simpler space, the moduli space of n-pointed stable genus zero curves $\overline{\mathbf{M}}_{0,n}$. As usual let D denote the boundary component (a normal crossing divisor) of $\overline{\mathbf{M}}_{0,n}$. We are interested in the finite generation of the ring $R = \bigoplus_{n\geq 0} \Gamma(n(K_{\overline{\mathbf{M}}_{0,n}} + \alpha \mathrm{D}))$ of sections associated to the divisor $K_{\overline{\mathbf{M}}_{0,n}} + \alpha D$. By Kawamata's basepoint freeness theorem (below), the associated sheaf to $n(K_{\overline{\mathbf{M}}_{0,n}} + \alpha \mathrm{D})$ is generated by its global sections, for

divisible $n >> 0$ and $0 < \alpha < 1$, if $K_{\overline{\mathbf{M}}_{0,n}} + \alpha D$ is nef and big. Thus R is finitely generated in such cases. For small α, however, $K_{\overline{\mathbf{M}}_{0,n}} + \alpha D$ is not nef. To get around this we will try to find nef and big divisors which have the same ring of sections as our log canonical rings. Moreover we will try to do this in such a way that we get an explicit description of our log canonical models in addition to existence.

In the next two sections we will relate the section rings of $K_{\overline{\mathbf{M}}_{0,n}} + \alpha D$ to certain divisors on the moduli spaces of weighted stable pointed curves. In particular positivity of these new divisors will imply that the weighted stable curve space is our log canonical model. In section 4 we will state the main theorem and show that Fulton's conjecture implies the positivity results which we need and section 5 will then survey the cases in which the result can be proved directly without Fulton's conjecture. Finally, in section 6, we examine the neighborhood of the $\mathfrak{s}_n$-equivariant ample cone on $\overline{\mathbf{M}}_{0,n}$ in which the divisors we are interested in live and show that it (conjecturally) has a very simple structure.

2   Preliminaries

We first recall some general facts.

Theorem 2.1 (Kawamata basepoint freeness)**.** *([KM], Theorem 3.3)*
Let (X, D) be a proper pair with klt singularities such that D is effective. Suppose A is a nef cartier divisor such that $A - \epsilon(K_X + D)$ is nef and big for some $\epsilon > 0$. Then $|nA|$ is basepoint free for $n >> 0$.

In particular we may have $A = K_X + D$.

Lemma 2.2. *Let $f : X \to Y$ be a proper morphism such that $f_* O_X = O_Y$ and let L be a locally free sheaf on Y. Then we have an isomorphism of global sections $\Gamma(X, f^*L) = \Gamma(Y, L)$.*

Proof. We have $\Gamma(X, f^*L) = \Gamma(Y, f_* f^*L)$ which, by the projection formula, is isomorphic to $\Gamma(Y, f_* O_X \otimes L)$. By assumption this is $\Gamma(Y, O_Y \otimes L) = \Gamma(Y, L)$. $\blacksquare$

We will be interested in the following application.

Proposition 2.3. *Let $f : X \to Y$ be a birational morphism between normal projective varieties. Let D be a divisor on Y and F be an effective divisor on X whose support is contracted by f. Then $\Gamma(X, f^*D + F) = \Gamma(Y, D)$.*

Note: Throughout the paper f^*D will be a shorthand for $f^* O_X(D)$.

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Proof. By the lemma we have $\Gamma(Y, D) = \Gamma(X, f^*D)$. Let U be the set on which f is an isomorphism. We note that the $codim(Y, Y/U) \geq 2$. In particular this implies that sections of D over U lift to sections of D over Y, so $\Gamma(Y, D) = \Gamma(U, D_{|U})$. Putting this all together we get a sequence of morphisms

$$\Gamma(X, f^*D) \hookrightarrow \Gamma(X, f^*D + F) \hookrightarrow \Gamma(U, (f^*D + F)_{|U}))$$

$$= \Gamma(U, f^*D_{|U}) = \Gamma(U, D_{|U}) = \Gamma(Y, D) = \Gamma(X, f^*D).$$

The first isomorphism is from the fact that F is supported away from U and the injections are the natural ones gotten by inclusion and restriction. Finally, these are all finite dimensional vector spaces so the fact that the total string of injective maps makes an isomorphisms implies that each individual map is an isomorphism.

$\square$

Finally we note that if D is ample in the above situation then proposition 2.3 implies that

$$Proj \bigoplus_{n \geq 0} \Gamma(X, n(f^*D + F)) = Proj \bigoplus_{n \geq 0} \Gamma(Y, nD) = Y$$

3   Weighted Stable Curves

$K_{\overline{\mathbf{M}}_{0,n}} + \alpha D$ can be written in a nice way in terms of the boundary divisors D_j, $2 \leq j \leq \lfloor n/2 \rfloor$, parameterizing nodal curves with a node separating j marked points from the others. In particular we have:

Lemma 3.1. *([Pan], Proposition 1)*

$$K_{\overline{\mathbf{M}}_{0,n}} = \sum_{j=2}^{\lfloor n/2 \rfloor} D_j \left(\frac{(j-2)(n-1) - j(j-1)}{n-1} \right), D = \sum_{j=2}^{\lfloor n/2 \rfloor} D_j.$$

This implies that $K_{\overline{\mathbf{M}}_{0,n}} + \alpha D$ is effective (big) when $\alpha \geq \frac{2}{n-1}(>)$. [KeM], Lemma 4.8 implies that the inequality for bigness is tight. Moreover since D is a normal crossing divisor on a smooth variety this implies by definition of log canonical/terminal that $K_{\overline{\mathbf{M}}_{0,n}} + \alpha D$ is log canonical (terminal) for $\alpha \leq 1(< 1)$. From now on we will assume α is a rational number in the interval of interest,

$$\frac{2}{n-1} < \alpha \leq 1.$$

Definition 3.2. For a given n and a given number $\alpha \in (\frac{2}{n-1}, 1]$, let k be the largest integer in the range $1, \ldots, \lfloor n/2 \rfloor$ such that $\alpha \leq 2/(k+1)$. We define the following $\mathfrak{s}_n$-equivariant divisor on $\overline{\mathbf{M}}_{0,n}$:

$$A_\alpha := K_{\overline{\mathbf{M}}_{0,n}} + \sum_{j=2}^{k} D_j \left(\binom{j}{2} \alpha - (j-2) \right) + \sum_{j=k+1}^{\lfloor n/2 \rfloor} \alpha D_j.$$

These will be the divisors we are interested in. As we will see shortly, they are pullbacks of nice log canonical divisors on certain related moduli spaces. It should also be noted that it makes since to allow $\alpha = \frac{2}{n-1}$ but it is uninteresting as $A_{\frac{2}{n-1}} = 0$.

Let $\overline{\mathbf{M}}_{0,\mathcal{A}}$ be the moduli space of weighted stable pointed curves associated to n points with symmetric weight $\mathcal{A} = \{a, a, \ldots, a\}$ with $\frac{1}{k+1} < a \leq \frac{1}{k}$. This space is smooth and projective. In his introduction to these spaces, Hassett [Ha] showed that there is a natural birational morphism $\rho : \overline{\mathbf{M}}_{0,n} \to \overline{\mathbf{M}}_{0,\mathcal{A}}$. In particular this map, on the level of pointed stable genus zero curves, acts on curves by contracting components which become unstable with the new weights. i.e. if a component has b nodes and c marked points then it will be contracted unless $ac + b > 2$. Our map ρ is an isomorphism away from the boundary D and contracts precisely the boundary components D_i with $i \neq 2$ and $ia \leq 1$.

From the properties of the map ρ described in [Ha], section 4, we have the following:

Lemma 3.3. *The map ρ factors into a sequence of blowups along smooth centers*

$$\rho : \overline{\mathbf{M}}_{0,n} \overset{B_3}{\to} \overline{\mathbf{M}}_{0,A(3)} \overset{B_4}{\to} \overline{\mathbf{M}}_{0,A(4)} \to \ldots \overset{B_k}{\to} \overline{\mathbf{M}}_{0,A(k)} = \overline{\mathbf{M}}_{0,\mathcal{A}}$$

where map B_i blows down the image of D_i under the map $B_{i-1} \circ B_{i-2} \circ \cdots \circ B_3$. If $k < 3$, ρ is an isomorphism. Here $A(i) = \{a, a, \ldots, a\}$ with $\frac{1}{i+1} < a \leq \frac{1}{i}$.

Theorem 3.4. *Let E denote the boundary of $\overline{\mathbf{M}}_{0,\mathcal{A}}$. Then $\rho^*(K_{\overline{\mathbf{M}}_{0,\mathcal{A}}} + \alpha E) = A_\alpha$ and $K_{\overline{\mathbf{M}}_{0,n}} + \alpha D - A_\alpha$ is effective and ρ-exceptional.*

Proof. Let D_j^i be the image of D_j under the map $B_{i-1} \circ B_{i-2} \circ \cdots \circ B_3$. ρ contracts precisely $D_3, \ldots, D_k$ and is birational on $D_2, D_{k+1}, \ldots D_{\lfloor n/2 \rfloor}$ so we have a discrepancy equation of the form

$$K_{\overline{\mathbf{M}}_{0,n}} = -\rho_*^{-1}(\alpha E) + \rho^*(K_{\overline{\mathbf{M}}_{0,\mathcal{A}}} + \alpha E) + \sum_{i=3}^{k} a_i D_i$$

$$= -\alpha(D_2 + D_{k+1} + \ldots D_{\lfloor n/2 \rfloor}) + \rho^*(K_{\overline{\mathbf{M}}_{0,n,\alpha}} + \alpha E) + \sum_{i=3}^{k} a_i D_i.$$

We can use our factorization of ρ to compute the a_i via the discrepancy formula for a blow-up. Suppose we've already done this for $i + 1 \leq j \leq k$. $M_{0,A(i-1)} \overset{B_i}{\to} M_{0,A(i)}$ is the blow-up map along D_i^i. Now we know that D_i is canonically isomorphic to a finite union of copies of $\overline{M}_{0,i+1} \times \overline{M}_{0,n-i+1}$ and the image of ρ is birational to a union of copies of $\overline{M}_{0,n-i+1}$. In particular, D_i^i has codimension $(i+1) - 3 + 1 = i - 1$. Therefore, calculating the discrepancy from the blow-up maps, we get

$$a_i = (i-1) - 1 - \sum_{j=2,i+1,i+2\ldots\lfloor n/2 \rfloor} \beta_j \, mult_{D_i^i} D_j^i$$

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where β_j equals a_i for $i+1 \le j \le k$ and α otherwise. That is to say they are the parts of the discrepancy formula that we have already calculated.

We break symmetry in order to calculate the needed multiplicities. Let δ_S, where S is a subset of $\{1, \ldots, n\}$, represent the standard divisors of pointed curves separating S and S^c by a node. Let $S = \{1, \ldots, i\}$ and T be a subset of $\{1, \ldots, n\}$ of order j. We can calculate $mult_{D_i^i} D_j^i$ by computing multiplicities on the images of δ_S and the $\delta_T{}'s$.

By Keel's description of the intersection ring in [Ke], δ_S and δ_T intersect iff $S \subset S'$ or $S \subset T'$ or $T \subset S'$ or $T \subset T'$. $mult_{\delta_S}\delta_T$ equals one in cases in which the two divisors intersect.

In the cases that $S \subset T$ or $S \subset T^c$ the intersection $\delta_S \cap \delta_T$ will be contracted by our map $B_{i-1} \circ B_{i-2} \circ \cdots \circ B_3$. The corresponding multiplicity $mult_{\text{image}(\delta_S)}\text{image}(\delta_T)$ becomes zero in this case.

The other possibility for intersection is $S^c \subset T$ or $S^c \subset T^c$. In either of these cases we are forced to have less elements in T than in S. However the order j of T is either 2 or $\ge i+1$ by assumption. Therefore we must have $j = 2$. In this case the intersection does not get contracted, and the multiplicity of the images of the two divisors is still one. To go back to the symmetric case we observe that there are $\binom{j}{2}$ divisors $\delta_{\{a,b\}}$ where $\{a,b\} \subset \{1, \ldots, j\}$. Thus the multiplicity of δ_S with D_2 is $\binom{j}{2}$. Since S was arbitrary the multiplicity of D_i with D_2 is the same and as observed the multiplicities of their images are also the same. Now we have:

$$K_{\overline{\mathbf{M}}_{0,n}} = -\alpha(D_2 + D_{k+1} + \ldots D_{\lfloor n/2 \rfloor}) + \rho^*(K_{\overline{\mathbf{M}}_{0,n,\alpha}} + \alpha D) - \sum_{i=3}^{k} (\binom{j}{2}\alpha - (j-2))D_i.$$

Rearranging the above equation gives us A_α as wished. Also $K_{\overline{\mathbf{M}}_{0,n}} + \alpha D - A_\alpha = K_{\overline{\mathbf{M}}_{0,n}} + \alpha D - (K_{\mathbf{M}_{0,n}} + \sum_{j=1}^{k} D_j \left(\binom{j}{2}\alpha - (j-2)\right) + \sum_{j=k+1}^{\lfloor n/2 \rfloor} \alpha D_j = \sum_{i=3}^{k}((j-2) + \alpha(1 - \binom{j}{2}))$ which is ρ-exceptional by construction and is effective since $\alpha \le \frac{2}{k+1} \le \frac{2}{j+1}$, so that the coefficients of our sum are greater than or equal to $\frac{1}{j+1}((j-2)(j+1) + 2 - j(j-1)) = 0$. $\square$

Corollary 3.5. *If $K_{\overline{\mathbf{M}}_{0,\mathcal{A}}} + \alpha E$ is ample then $\overline{\mathbf{M}}_{0,\mathcal{A}}$ is the log canonical model of $\overline{\mathbf{M}}_{0,n}$ associated to the divisor $K_{\overline{\mathbf{M}}_{0,n}} + \alpha D$. In particular this will be true if A_α is nef and only contracts ρ-exceptional curves.*

Note: Technically $\overline{\mathbf{M}}_{0,\mathcal{A}}$ is only defined for $na > 2$. Therefore for the smallest values of α the above process was not well defined. However, as mentioned in [Ha], we can define $\overline{\mathbf{M}}_{0,\mathcal{A}}$ for $a \le 2/n$ to be $\mathbb{P}^1/SL_2$ with symmetric weight a. There still exists a map $\rho : \overline{\mathbf{M}}_{0,n} \to \mathbb{P}^1/SL_2$ which

contracts all the boundary divisors except for D_2. The above computation plays out unchanged and the pullback result is still true.

4   F-divisors

Definition 4.1. Let D be a divisor on $\overline{M}_{0,n}$. We say that D is *F-nef* (or Fulton nef) if it intersects all vital curves on $\overline{M}_{0,n}$ nonnegatively, where by a vital curve we mean an irreducible component of the closed set in $\overline{M}_{0,n}$ corresponding to pointed genus zero stable curve with at least n-4 nodes.

Figure 1: The generic element of a vital curve in $\overline{M}_{0,10}$
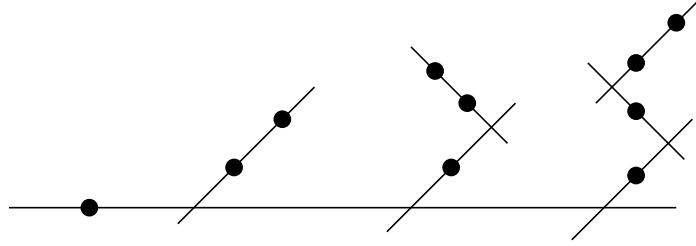
We denote the closure of the real cone of effective curves up to numerical equivalence by $\overline{NE}_1$.

Conjecture 4.2. *Fulton's Conjecture*

$\overline{NE}_1$ *is generated by vital curves. In other words, a divisor is nef (ample) if and only if it intersects all vital curves nonnegatively (positively).*

Theorem 4.3 (Farkas, Gibney). *[FaGi, Gi] Fulton's conjecture is true for $\mathfrak{s}_n$-equivariant divisors on $\overline{M}_{0,n}$ for $n \leq 24$.*

Any vital curve C is represented by a partition $a+b+c+d = n$. Specifically the n-pointed stable curve corresponding to the generic point of C has n-3 components and therefore exactly one component with 4 'special points'-marked points or nodes-which partition the marked points into four sets. Figure 1, for example, corresponds to the partition $1+2+3+4 = 10$. The intersections with such a curve are well known:

Theorem 4.4 (Keel, McKernan). *[KeM] If a divisor equals $\sum_{j=2}^{\lfloor n/2 \rfloor} r_j D_j$ than it is F-nef if and only if*

$$r_{a+b} + r_{a+c} + r_{a+c} - r_a - r_b - r_c - r_d \geq 0$$

for all partitions $a + b + c + d = n$. Here $r_i = r_{n-i}$ and $r_1 = 0$.

We are now in position to prove our main result.

Theorem 4.5 (Main Theorem).
Fix $n \geq 4$ and α a rational number in $(\frac{2}{n-1}, 1]$. Assume the ($\mathfrak{s}_n$-equivariant) Fulton conjecture. Let $\overline{M}_{0,n}(\alpha)$ denote the log canonical model of $\overline{M}_{0,n}$ with respect to $K + \alpha D$. If α is in the range $(\frac{2}{k+2}, \frac{2}{k+1}]$ for some $k = 1, ..., \lfloor \frac{n-1}{2} \rfloor$ then $\overline{M}_{0,n}(\alpha) \simeq \overline{M}_{0,(1/k,...,1/k)}$. If α is in the range $(\frac{2}{n-1}, \frac{2}{\lfloor n/2 \rfloor + 1}]$ then $\overline{M}_{0,n}(\alpha) \simeq (P^1)^n // SL_2$.

Applying corollary 3.5 we will prove the main theorem by showing that the A_α divisors are F-nef.

Proposition 4.6. A_α *is F-nef for all* n, α.

Using the above criterion, we need only check that the coefficients of A_α satisfy the inequalities for each partition. We observe that if α is in $[2/(k+2), 2/(k+1)]$ then A_α is a convex sum of $A_{2/(k+2)}$ and $A_{2/(k+2)}$:

$$\alpha = \frac{2t}{k+2} + \frac{2(t-1)}{k+1} \text{ implies } A_\alpha = tA_{2/(k+2)} + (1-t)A_{2/(k+2)}.$$

Therefore to prove that intersections are nonnegative we only need to check α of the form $\frac{2}{k+1}$. Henceforth let $\alpha = \frac{2}{k+1}$. By theorems 5.1 and 5.2 we need only check α for $2 \leq k \leq \lfloor n/2 \rfloor - 1$.

Lemma 4.7. *Let $n \geq 6$ and $1 < k < \lfloor n/2 \rfloor$ be arbitrary. Set $\alpha = \frac{2}{k+1}$ and define the following functions:*

$$f(j) = \frac{j(j-1)\left((n-1)\frac{\alpha}{2} - 1\right)}{n-1}, \quad g(j) = \frac{(n-1)(\alpha + j - 2) - j(j-1)}{n-1}.$$

Then, for any sum of positive integers $a + b + c + d = n$ the function

$$h(j) = \begin{cases} f(j), & \text{for } 1 \leq j \leq k; \\ g(j), & \text{for } k < j < n - k; \\ f(n-j), & \text{for } n - k \leq j \leq n \end{cases} \tag{1}$$

satisfies $h(a + b) + h(a + c) + h(a + d) - h(a) - h(b) - h(c) - h(d) \geq 0$.

Proof. The function $h(a + b) + h(a + c) + h(a + d) - h(a) - h(b) - h(c) - h(d)$ depends heavily on what the value of k is with respect to n, a, b, c and d. For a given n, a, b, c, d with $a \leq b \leq c \leq d$ we will partition the possible values of k into subsets on which h acts 'nicely'. Specifically, we will look at the following partition into cases. The fact that some of these cases may not exist for certain values of a, b, c, d will be dealt with as the individual cases are examined.

1. $a > k$ and $a \leq b \leq c \leq d$ as always.

2. $a \leq k, b > k$

3. $b \le k, c > k, a + b > k$

4. $b \le k, c > k, a + b \le k$

5. $c \le k, d > k, a + b > k$

6. $c \le k, d > k, a + b \le k, a + c > k$

7. $c \le k, d > k, a + c \le k, b + c > k$

8. $c \le k, d > k, b + c \le k, a + b + c > k$

9. $c \le k, d > k, b + c \le k, a + b + c \le k$

10. $d \le k, a + b > k$

11. $d \le k, a + b \le k, a + c > k$

12. $d \le k, a + c \le k, a + d > k, b + c > k$

13. $d \le k, a + c \le k, a + d \le k, b + c > k$

14. $d \le k, a + c \le k, a + d > k, b + c \le k$

15. $d \le k, a + c \le k, a + d \le k, b + c \le k$

These cases are all-inclusive. We will now show that $h(a + b) + h(a + c) + h(a + d) - h(a) - h(b) - h(c) - h(d)$ is nonnegative for each of these subsets. In all cases, the inequality comes easily from the inequalities in the given cases. We will skim the details since they are easy and repetitive.

case 1) In this case, the inequalities show us that $h(a + b) + h(a + c) + h(a + d) - h(a) - h(b) - h(c) - h(d)$ can be written as $g(a + b) + g(a + c) + g(a + d) - g(a) - g(b) - g(c) - g(d)$. This is because all the values $a, b, c, d, a + b, a + c, a + d$ are by assumption greater than k and they are all less than $n - k$ by observation. For example $a + d \le n - k$ since $b > k$ and $c > k$ and $a + b + c + d = n$. Using this we get the calculation(after some simplification)

$$h(a + b) + h(a + c) + h(a + d) - h(a) - h(b) - h(c) - h(d)$$
$$= g(a + b) + g(a + c) + g(a + d) - g(a) - g(b) - g(c) - g(d)$$
$$= \frac{2k}{1 + k}$$

which is clearly positive.

case 2) Under the given hypotheses we get $h(a+b)+h(a+c)+h(a+d)-h(a)-h(b)-h(c)-h(d) = g(a + b) + g(a + c) + g(a + d) - f(a) - g(b) - g(c) - g(d)$. This gives us

$$h(a + b) + h(a + c) + h(a + d) - h(a) - h(b) - h(c) - h(d) = \frac{a(2 + k - a)}{1 + k}$$

which is again nonnegative since $k \geq a$ in this case.

case 3) In this case we have

$$h(a+b) + h(a+c) + h(a+d) - h(a) - h(b) - h(c) - h(d)$$
$$= g(a+b) + g(a+c) + g(a+d) - f(a) - f(b) - g(c) - g(d)$$
$$= \frac{(b-2)(k-b) + a(2+k-a)}{1+k}.$$

The second half of this is positive since $k \geq a$. The first half is similarly positive unless $b = 1$. but that would imply $a = 1$ and thus $2 = a + b > k$ which does not hold. Thus the whole equation is positive.

case 4) Here we have

$$h(a+b) + h(a+c) + h(a+d) - h(a) - h(b) - h(c) - h(d)$$
$$= f(a+b) + g(a+c) + g(a+d) - f(a) - f(b) - g(c) - g(d)$$
$$= \frac{2ab}{1+k} > 0.$$

case 5)

$$h(a+b) + h(a+c) + h(a+d) - h(a) - h(b) - h(c) - h(d)$$
$$= g(a+b) + g(a+c) + g(a+d) - f(a) - f(b) - f(c) - g(d)$$
$$= \frac{(k-c)(c-2) + (k-b)(b-2) + a(2+k-a)}{1+k}$$

which is positive except potentially when $b = 1$. However, $a = b = 1$ is not possible in this case.

case 6)

$$h(a+b) + h(a+c) + h(a+d) - h(a) - h(b) - h(c) - h(d)$$
$$= f(a+b) + g(a+c) + g(a+d) - f(a) - f(b) - f(c) - g(d)$$
$$= \frac{2ab + (c-2)(k-c)}{1+k}$$

which is positive because $k \geq c$ and $a + c > k$ so that $c \geq 2$.

case 7)

$$h(a+b) + h(a+c) + h(a+d) - h(a) - h(b) - h(c) - h(d)$$
$$= f(a+b) + f(a+c) + g(a+d) - f(a) - f(b) - f(c) - g(d)$$
$$= \frac{a(a+2b+2c-2-k)}{1+k} > 0 \text{ since } b + c > k.$$

case 8)

$$h(a + b) + h(a + c) + h(a + d) - h(a) - h(b) - h(c) - h(d)$$
$$= f(a + b) + f(a + c) + f(b + c) - f(a) - f(b) - f(c) - g(d)$$
$$= \frac{(a + b + c - 2)(a + b + c - k)}{1 + k} > 0 \text{ since } a + b + c > k.$$

case 9)

$$h(a + b) + h(a + c) + h(a + d) - h(a) - h(b) - h(c) - h(d)$$
$$= f(a + b) + f(a + c) + f(b + c) - f(a) - f(b) - f(c) - f(a + b + c)$$
$$= 0$$

case 10)

$$h(a + b) + h(a + c) + h(a + d) - h(a) - h(b) - h(c) - h(d)$$
$$= g(a + b) + g(a + c) + g(a + d) - f(a) - f(b) - f(c) - f(d)$$
$$= \frac{(k - d)(d - 2) + (k - c)(c - 2) + (k - b)(b - 2) + a(2 + k - a)}{1 + k}$$

is positive by the identical argument as in case 5.

case 11)

$$h(a + b) + h(a + c) + h(a + d) - h(a) - h(b) - h(c) - h(d)$$
$$= f(a + b) + g(a + c) + g(a + d) - f(a) - f(b) - f(c) - f(d)$$
$$= \frac{2ab + (c - 2)(k - c) + (d - 2)(k - d)}{1 + k} > 0 \text{ since } a + c > k, d \le k.$$

case 12)

$$h(a + b) + h(a + c) + h(a + d) - h(a) - h(b) - h(c) - h(d)$$
$$= f(a + b) + f(a + c) + g(a + d) - f(a) - f(b) - f(c) - f(d)$$
$$= \frac{a(2b + 2c + a - 2 - k) + (d - 2)(k - d)}{1 + k} > 0 \text{ since } b + c > k, d \ge 2, k \ge d.$$

case 13)

$$h(a+b) + h(a+c) + h(a+d) - h(a) - h(b) - h(c) - h(d)$$
$$= f(a+b) + f(a+c) + f(a+d) - f(a) - f(b) - f(c) - f(d)$$
$$= \frac{2a(n-2-k)}{1+k} > 0.$$

case 14)

$$h(a+b) + h(a+c) + h(a+d) - h(a) - h(b) - h(c) - h(d)$$
$$= f(a+b) + f(a+c) + f(b+c) - f(a) - f(b) - f(c) - f(d)$$
$$= \frac{(n-2d)(n-2)}{1+k} > 0 \text{ since } d \le k < \lfloor n/2 \rfloor.$$

case 15) This last case can't actually happen with the assumption $k < \lfloor n/2 \rfloor$. This concludes the proof of our lemma. □

We note that in our intersection calculation we got a strictly positive number in every case except when $a + b + c \le k$. This is exactly what we would expect since those values represent precisely the curves which get contracted under the map $\rho : \overline{\mathbf{M}}_{0,n} \to \overline{\mathbf{M}}_{0,\mathcal{A}}$.

Proof of Proposition 4.6.
By applying Theorem 4.4 we have proved that A_α intersects nonnegatively with all vital curves in the cases when $\alpha = \frac{2}{k+1}$, $2 \le k \le \lfloor n/2 \rfloor - 1$ and $n \ge 6$. Since divisors that are nef are certainly also F-nef, we have shown that A_α is F-nef for all $\frac{2}{n-1} \le \alpha \le 1$, $n \ge 6$. The result is trivial for $n < 6$.

□

5 Partial results

Certain ranges of A_α can be explicitly shown to be nef.

Theorem 5.1. *For $\frac{2}{n-1} < \alpha \le \frac{2}{\lfloor n/2 \rfloor + 1}, A_\alpha$ is semi-ample.*

Proof. This range corresponds exactly to when our map ρ is of the form $\rho : \overline{\mathbf{M}}_{0,n} \to \mathbb{P}^1/SL_2$. Because we have contracted all the rays corresponding to D_j, $j \ne 2$ of $\overline{\mathbf{M}}_{0,n}$, the $\mathfrak{s}_n$-equivariant Picard group of $\mathbb{P}^1/SL_2$ is one dimensional and generated by the image of D_2. Also the canonical class is simply

$$K = \rho_*(K_{\overline{\mathbf{M}}_{0,n}}) = \rho_*(\frac{-2}{n-1}D_2 + \rho - \text{exceptional divisors}) = \frac{-2}{n-1}[\text{image of } D_2].$$

Therefore $K + \alpha E = (\alpha - \frac{2}{n-1})[\text{image of } D_2]$ is ample for our range of α which implies A_α is the pullback of an ample divisor as we wanted.

□

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Theorem 5.2 (KM). *A_α is ample for $2/3 < \alpha \le 1$*

Keel and McKernan actually prove this for $A_1 = K_{\overline{\mathbf{M}}_{0,n}} + D$. However $A_\alpha = K_{\overline{\mathbf{M}}_{0,n}} + \alpha D$ in these cases and the proof generalizes word for word.
Keel and McKernan also developed a combinatorial tool that helps prove in special cases that an F-nef divisor is actually nef.

Theorem 5.3 (KeM). *Let R be an extremal ray of the cone $\overline{NE}_1$. Supposed D is a $\mathbb{Q}$-divisor with $D - G$ effective and $(K_{\overline{\mathbf{M}}_{0,n}} + G).R \le 0$. Then R is spanned by a vital curve.*

Corollary 5.4. *If $K_{\overline{\mathbf{M}}_{0,n}} + G$ as above is F-nef then it is nef.*

Proof. Otherwise, we'd have some R with $(K_{\overline{\mathbf{M}}_{0,n}} + G).R < 0$. But that would imply that $K_{\overline{\mathbf{M}}_{0,n}} + G$ intersects a vital curve negatively which is a contradiction. $\square$

Corollary 5.5. *A_α is nef for $\alpha \ge \frac{1}{3}$.*

Let $\alpha = \frac{2}{k+1}$ and $n > 13$.

Proof. It is enough to show that some positive multiple of A_α is nef. Let $c > 0$ and set $B = cA_\alpha - K_{\overline{\mathbf{M}}_{0,n}} =$

$$\sum_{j=1}^{k} D_j \left(\binom{j}{2} c\alpha - (j-2) - \frac{j(j-1)}{n-1} \right) + \sum_{j=k+1}^{\lfloor n/2 \rfloor} D_j \left(c(j-2+\alpha) - (j-2) - \frac{j(j-1)}{n-1} \right)$$

If we can find a value of c so that each coefficient is between zero and one, then we can apply the proposition.

By examining the lower terms we immediately are forced to have the inequalities:

$$c \ge \frac{k+1}{n-1} + \frac{k+1}{6} \quad \text{and} \quad c \le \frac{k+1}{n-1} + \frac{k+1}{k}.$$

Therefore this only has a chance of satisfying our condition for $k \le 6$. Similarly we get the following inequality from bounds on the upper terms:

$$c \ge \frac{(\lfloor n/2 \rfloor - 2)(n + \lfloor n/2 \rfloor) + 2}{(n-1)(\lfloor n/2 \rfloor - 2 + \alpha)} \quad \text{and} \quad c \le \frac{k(n-k)}{(n-1)(k-1+\alpha)}.$$

For $k \le 6$ we can simply plug k into these 4 inequalities and we see that they are all satisfied for $k \le 5$ but not for $k = 6$.

For higher values of k we may at least hope to make all terms positive and some terms ≤ 1. We can do this, but the inequalities still force most coefficient of the D_j's to be quite large. $\square$

6   The Fulton Cone

One way to prove that our A_α's are nef would be to prove Fulton's conjecture. We of course don't need quite so strong a result and in fact, from computational data, it appears that at least some of the A_α's sit in a very nice part of the Fulton cone of F-nef divisors. Although the major combinatorial techniques don't appear to be of use, it would be nice if this corner of the Fulton cone could be proven to be nef by geometric means.

Notation 6.1. Let $W=\mathbb{P}(\mathbb{Q}^{\lfloor n/2 \rfloor - 1})$ be the projectivized cone of $\mathfrak{s}_n$-equivariant divisors of $\overline{M}_{0,n}$ with standard basis $D_2, \ldots D_{\lfloor n/2 \rfloor}$ and corresponding coordinates $(r_2 \ldots r_{\lfloor n/2 \rfloor})$. Let F be the subcone of F-nef divisors.

For each sum $a+b+c+d = n$ we have an associated hyperplane in W. We will denote the 'special' hyperplanes corresponding to $1+1+i+(n-i-2) = n$ by V_i. Other hyperplanes will simply be denoted $V(a,b,c,d)$.

We will be studying the portion of F which contains A_α for small α.

Proposition 6.1. *For $2 \le k \le \lfloor n/2 \rfloor$, the minimal face of F containing $A_{\frac{2}{k+1}}$ is*

$$F_k = V_1 \cap V_2 \cap \ldots \cap V_{k-2} \cap F \subset W.$$

In particular $A_{\frac{2}{k+1}}$ is in the interior of a face of (projective) dimension $\lfloor n/2 \rfloor - k$ whose boundary contains A_α for $\alpha \le \frac{2}{k+2}$.

Here $F_2 = F$ and $F_{\lfloor n/2 \rfloor}$ is a (projective) vertex of F equal to $A_{\frac{2}{\lfloor n/2 \rfloor + 1}}$ (which is in fact nef unconditionally by Theorem 5.1).

Proof. By the proof of Theorem 4.6 $A_{\frac{2}{k+1}}$ contracts precisely vital curves corresponding to partitions $a+b+c+d = n$ such that $a+b+c \le k$. Therefore the minimal face of F containing $A_{\frac{2}{k+1}}$ is cut out by $\bigcap_{a+b+c \le k} V(a,b,c,d)$.

Claim:
$$\bigcap_{a+b+c \le k} V(a,b,c,d) = V_1 \cap V_2 \cap \ldots \cap V_{k-2}.$$

We certainly have $\bigcap_{a+b+c \le k} V(a,b,c,d) \subset V_1 \cap V_2 \cap \ldots \cap V_{k-2}$. To prove the other inclusion we give an explicit description of $V_1 \cap V_2 \cap \ldots \cap V_{k-2}$.

$V_1 \cap V_2 \cap \ldots \cap V_{k-2}$ is defined by equations $r_2 + 2r_{i+1} - r_i - r_{i+2} = 0$, or $r_{i+2} = r_2 + 2r_{i+1} - r_i$ for $i \le k-2$. The only solutions up to multiplication by a constant for this recurrence relation are $r_i = \binom{j}{2}$ for $2 \le i \le k$ and r_i free for $k < i \le \lfloor n/2 \rfloor$.

Now for $a+b+c \le k \le \lfloor n/2 \rfloor$, $V(a,b,c,d)$ has defining equation $r_{a+b} + r_{a+c} + r_{b+c} - r_a - r_b - r_c - r_{a+b+c} = 0$. To prove the claim we need only show that any such equation is satisfied by all points of W with $r_i = \binom{j}{2}$ for $i \le k$.

This condition is equivalent to $\binom{a+b}{2} + \binom{a+c}{2} + \binom{b+c}{2} - \binom{a}{2} - \binom{b}{2} - \binom{c}{2} - \binom{a+b+c}{2} = 0$ which is always true. This proves our claim.

The claim immediately implies the first statement of the proposition. Additionally, we showed in the proof of the claim that the dimension of $V_1 \cap V_2 \cap \ldots \cap V_{k-2}$ is $(\lfloor n/2 \rfloor - 2) - (k-2) = \lfloor n/2 \rfloor - k$. Since each A_α is in F, this completes the proof.

$\square$

The following is a description of the faces $F_{\lfloor n/2 \rfloor -1}$ and $F_{\lfloor n/2 \rfloor -2}$. Specifically, for each n, we make a table with the following information: the first column lists the hyperplanes which cut out the vertices of the (1-dimensional) face $F_{\lfloor n/2 \rfloor -1}$, the second column describes the shape of $F_{\lfloor n/2 \rfloor -2}$ and the last column lists the hyperplanes which cut out edges of $F_{\lfloor n/2 \rfloor -2}$.

n	facet inducing hyperplanes of $F_{\lfloor n/2 \rfloor -1}$	# of facets of $F_{\lfloor n/2 \rfloor -2}$	facet inducing hyperplanes of $F_{\lfloor n/2 \rfloor -2}$
6	V_1 and V_2		
7	V_1 and $V(1,2,2,2)$		
8	V_2 and V_3	4	$V_1, V_2, V_3, V(2,2,2,2)$
9	V_2 and V_3	4	$V_1, V_2, V_4, V(2,2,2,3)$
10	V_3 and V_4	4	$V_2, V_3, V_4, V(1,3,3,3)$
11	V_3 and V_4	4	$V_2, V_3, V_4, V(2,3,3,3)$
12	V_4 and V_5	3	V_3, V_4, V_5
13	V_4 and V_5	4	$V_3, V_4, V_5, V(1,4,4,4)$
$n = 14, \ldots$	$V_{\lfloor n/2 \rfloor -2}, V_{\lfloor n/2 \rfloor -1}$	3	$V_{\lfloor n/2 \rfloor -3}, V_{\lfloor n/2 \rfloor -2}, V_{\lfloor n/2 \rfloor -1}$

Example 6.2. For $n = 8$, the faces $F_{\lfloor n/2 \rfloor}$ and $F_{\lfloor n/2 \rfloor -1}$ have already stabilized into the form that all such faces have for large n. $F_{\lfloor n/2 \rfloor -2}$, on the hand, has not yet stabilized into the expected form of a 3-simplex.
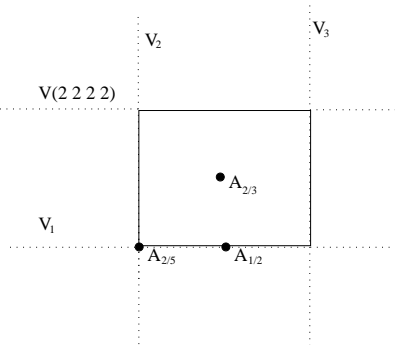

Figure 2:   The cone F for $\overline{\mathbf{M}}_{0,8}$

The structure of vertices, edges, etc. of the higher dimensions faces $F_{\lfloor n/2 \rfloor - t}$ stabilize into a similar pattern. In fact we can make this more precise.

Theorem 6.3 (Version 1). *The Fulton cone is stably simplicial in the neighbor of the A_α's in the sense that, given t, $F_{\lfloor n/2 \rfloor - t}$ is a simplex for $n >> 0$.*

Definition 6.4. By the *F-simplex* we denote the simplex in W induced by $V_1, \ldots, V_{\lfloor n/2 \rfloor - 1}$. Since the equations for these hypersurfaces are linearly independent this does indeed make a simplex. We denote the vertices by $p_{k+1} = \bigcap_{i \neq k} V_i$.

By Proposition 6.1 we know that F_k is contained in the face of the F-simplex cut out by $V_1 \cap V_2 \cap \ldots \cap V_{k-2}$. Such a face is of course again a simplex of smaller dimension. To prove the theorem we will show that this simplex is actually equal to F_k for sufficiently large k. It is enough to prove that the vertices p_k are $F - nef$ for large k.

Lemma 6.5. *The point p_k is represented by the divisor*

$$\frac{2}{n-1} \left[\sum_{i=2}^{k} \binom{i}{2} D_i + \sum_{i=k+1}^{\lfloor n/2 \rfloor} \left(\binom{i}{2} - (i-k)\left(\frac{n-1}{2}\right) \right) D_i \right]$$

$$= -K_{\overline{\mathbf{M}}_{0,n}} + \sum_{i=2}^{k} (i-2) D_i + \sum_{i=k+1}^{\lfloor n/2 \rfloor} (k-2) D_i$$

Proof of Lemma. The equation of V_i for $i \leq \lfloor n/2 \rfloor - 2$ is

$$r_{i+2} = r_2 + 2r_{i+1} - r_i.$$

Ignoring $V_{\lfloor n/2 \rfloor - 1}$ for the moment, we can solve these equations for $2 \leq i \leq k-2, k \leq i \leq \lfloor n/2 \rfloor - 2$. This gives us:

$$r_i = \binom{i}{2} \text{ for } i \leq k;$$

$$r_i = \left(\binom{i-k}{2} + (i-k)r_{k+1} - (i-k-1)\binom{k}{2} \right) \text{ for } k+1 \leq i \leq \lfloor n/2 \rfloor.$$

Depending on the parity of n, $V_{\lfloor n/2 \rfloor - 1}$ is cut out by $r_{n/2} = r_{n/2-1} - \frac{1}{2}$ or $r_{\lfloor n/2 \rfloor} = r_{\lfloor n/2 \rfloor - 1} - 1$. We solve for r_{k+1} by plugging in the values for $r_{\lfloor n/2 \rfloor}$ and $r_{\lfloor n/2 \rfloor - 1}$. Doing so in either the even or the odd case gives us $r_{k+1} = \binom{k+1}{2} - \frac{n-1}{2}$. This in turn implies that for $i \geq k+1$ we have

$$r_i = \binom{i-k}{2} + (i-k)(\binom{k+1}{2} - \frac{n-1}{2}) - (i-k-1)\binom{k}{2} = \binom{i}{2} - (i-k)\left(\frac{n-1}{2}\right).$$

$\square$

We can now explicitly compute the vertices of the F-simplex which are F-nef.

Theorem 6.6 (Version 2). *F_k is the face of the F-simplex spanned by $p_k, \ldots p_{\lfloor n/2 \rfloor}$ whenever $k \geq \frac{n}{3}$. This bound is tight.*

Proof. As usual we will look at curves C corresponding to a partition $a + b + c + d = n$ with $a \leq b \leq c \leq d$.

case 1) $a + b \geq k$.
In this case we have $p_k \bullet C =$

$$-2+(k-2)+(k-2)+(k-2)-\min(a-2,k-2)-\min(b-2,k-2)-\min(c-2,k-2)-\min(d-2,k-2).$$

Since $k \geq n/3$ this implies $a, b \leq k$, so in fact (canceling the two's) we have

$$p_k \bullet C = 3k - a - b - \min(c,k) - \min(d,k) \geq 3k - a - b - c - d = 3k - n \geq 0.$$

case 2) $a + d \leq k$.
by canceling terms we get

$$p_k \bullet C = (a+b) + (a+c) + (a+d) - a - b - c - d = 2a > 0.$$

case 3) $a + b < k, a + d > k$.

$$p_k \bullet C = (a+b)+\min(a+c,k)+\min(k,b+c)-a-b-\min(c,k)-\min(d,k,a+b+c)$$
$$= \min(a+c,k) + \min(k,b+c) - \min(c,k) - \min(d,k,a+b+c) \geq 0$$

These three cases are all-inclusive so we have that p_k is F-nef whenever $k \geq \frac{n}{3}$. For tightness, we refer to the following example. $\qquad\qquad\square$

Example 6.7. Let $n = 3 * l + p$ where $p \in \{1,2,3\}, l \geq 2$. We will look at $p_{\lfloor n/2 \rfloor - k}$ where $k = l$. Let C be represented by $p + l + l + l = n$. Then $\lfloor n/2 \rfloor - k \bullet C = -1 + 3(l-2) - 3(k-2) = -1 < 0$. Thus $p_{\lfloor n/2 \rfloor - k}$ is not F-nef is this case.

Conjecture 6.8 (Weak Fulton). *The face of the $\mathfrak{s}_n$-equivariant nef cone containing A_α for $\alpha \leq \frac{2}{\lceil n/3 \rceil + 1}$ is a simplex generated by $p_{\lceil n/3 \rceil}, \ldots p_{\lfloor n/2 \rfloor}$.*

Remark 6.9. The weak Fulton conjecture would not by itself prove our description of the log canonical models of $\overline{\mathbf{M}}_{0,n}$. It would however give us a strong partial result.

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7 References

[BCHM] C. Birkar, P. Cascini, C. Hacon, J. McKernan, *Existence of minimal models for varieties of log general type*, math.AG/0610203

[CoHa] M. Cornalba, J. Harris, *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Sci. cole Norm. Sup. (4) 21 (1988), no. 3, 455–475.

[CHS] I. Coskun,, J. Starr, J. Harris, *The ample cone of the Kontsevich moduli space*, Canad. J. Math., to appear.

[EH] D. Eisenbud J. Harris, *The Kodaira dimension of the moduli space of curves of genus ≥ 23*, Invent. Math. 90 (1987), no. 2, 359-387.

[Fa] G. Farkas, *Syzygies of curves and the effective cone of $\overline{M}_g$*, Duke Math. J. 135 (2006), no. 1, 53–98.

[FaGi] G. Farkas, A. Gibney, *The Mori cones of moduli spaces of pointed curves of small genus*, Trans. Amer. Math. Soc. 355 (2003), no. 3, 1183–1199.

[Gi] A. Gibney, *Numerical criteria for divisors on M_g to be ample*, math.AG/0312072.

[GKM] A. Gibney, S. Keel, I. Morrison, *Towards the ample cone of $\overline{M}_{g,n}$*, J. Amer. Math. Soc. 15 (2002), no. 2, 273–294.

[HaMu] J. Harris, D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. 67 (1982), no. 1, 23-88.

[Ha] B. Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. 173 (2003), no. 2, 316–352.

[HaMo] J. Harris, I. Morrison, *Slopes of effective divisors on the moduli space of stable curves*, Invent. Math. 99 (1990), no. 2, 321–355.

[HH] B. Hassett, D. Hyeon, *Log canonical models for the moduli space of curves: First divisorial contraction*,math.AG/0607477v1.

[Ke] S. Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*. Trans. Amer. Math. Soc. 330 (1992), no. 2, 545–574.

[KeM] S. Keel, J. McKernan, *Contractible Extremal Rays on $\overline{M}_{0,n}$*, math.AG/9607009.

[KM] J. Kollar, S. Mori. *Birational geometry of algebraic varieties*, Cambridge

Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998.

[Pan] R. Pandharipande, *The canonical class of* $\overline{M}_{0,n}(\boldsymbol{P}^r, d)$ *and enumerative geometry*, Internat. Math. Res. Notices (1997), no. 4, 173–186.